EXTREME POINT PROPERTIES OF CONVEX BODIES IN REFLEXIVE BANACH SPACES

BY

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ABSTRACT

A study is made of topological and cardinality properties of the set of extreme points of bounded closed convex sets with interior in reflexive Banach spaces. Some related results, and applications to earlier work, are obtained as corollaries.

0. Introduction. The starting point for this paper is the observation that the set ext U of extreme points of the unit ball U of a reflexive Banach space E must be uncountable. It follows that if E is separable, then ext U cannot be an isolated set in the norm topology (a result which is shown to fail for a certain nonseparable space). Another easy consequence of the first theorem is the fact that any convex body in E has uncountably many extreme points. Examples are given and a problem posed concerning similar questions for certain subsets of the extreme points (exposed points, strongly exposed points), and applications of the first theorem are made to questions which arose in studying extensions of compact operators [3]. A brief final section deals with certain convex sets having countably many extreme points.

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1. Extreme points of convex bodies in reflexive spaces.

THEOREM 1.1. *If E is an infinite dimensional reflexive Banach space, then the set* ext U *of extreme points of the unit ball U of E is uncountable.*

Proof. Suppose that $\operatorname{ext} U = \{x_n\}_{n=1}^{\infty}$ and for each n let

$$F_n = \{f \in E^*: \|f\| \leqq 1 \text{ and } |f(x_n)| = \|f\|\}.$$

It follows from the weak lower semicontinuity of the norm in E^* that each F_n is weakly closed. From the reflexivity of E and the Krein-Milman theorem appiied to U it follows that the weakly compact unit ball U^* of E^* is the union of the sets

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F_n, $n = 1, 2, \cdots$. By the Baire category theorem, at least one of the sets F_n (say F_1) has nonempty weak interior relative to U^*. Let f_0 be a relative weak interior point; without loss of generality we can assume that $\|f_0\| < 1$. Thus, there exist points $y_1, y_2, \cdots, y_n$ in E such that $f \in F_1$ whenever

$$(*) \qquad \|f\| \leqq 1 \text{ and } |(f - f_0)(y_i)| < 1, \qquad i = 1, 2, \cdots, n.$$

To obtain a contradiction from this, let $N = \{f \in E^* : f(y_i) = f_0(y_i),\ i = 1, 2, \cdots, n$ and $f(x_1) = f_0(x_1)\}$. Since E is infinite dimensional, this flat of finite codimension contains a line through f_0, which necessarily intersects the unit sphere in a point g, $\|g\| = 1$. Since $g \in N$, the condition (*) implies that $g \in F_1$, so $1 = \|g\| = |g(x_1)| = |f_0(x_1)| = \|f_0\|$, a contradiction.

If E is a Banach space over the *complex* numbers, then x is an extreme point of the unit ball U of E if and only if λx is an extreme point, for each $|\lambda| = 1$. Thus, the existence of one extreme point implies the existence of uncountably many. If, however, we define two extreme points x and y to be equivalent provided $x = \lambda y$ for some $|\lambda| = 1$, then it makes sense to ask whether U can have countably many equivalence classes of extreme points. The proof of Theorem 1.1 applies without change to show that if E is reflexive and infinite dimensional, then the answer is negative.

A *convex body* is a bounded closed convex set having nonempty interior.

COROLLARY 1.2. *If F is an infinite dimensional reflexive Banach space and if C is a convex body in F, then* ext C *is uncountable.*

Proof. Let $E = F \times R$, with the norm $\|(x, r)\| = \max(\|x\|, |r|)$. It is clear that E is reflexive. Define an equivalent norm on E by taking the new unit ball to be the convex hull of C_1 and $-C_1$, where $C_1 = \{(x, 1) : x \in C\}$. It follows that with this norm, E is still a reflexive Banach space. Since $\operatorname{ext} U = \operatorname{ext} C_1 \cup (-\operatorname{ext} C_1)$, the set $\operatorname{ext} C_1$ (hence $\operatorname{ext} C$) is countable.

Let us remark that Corollary 1.2 implies in particular that if K is a w compact countable set in an infinite dimensional Banach space E then the interior of $C = \overline{\operatorname{Conv} K}$ is empty. Indeed assume that the interior of C is not empty. Since C is w compact, by a well known result of Krein, it follows that E is reflexive. But since $\operatorname{ext} C \subset K$ we get a contradiction to Corollary 2.1.

If E is a *separable* reflexive Banach space, then every convex body C in E is compact, metrizable and separable (in the weak topology), so that $\operatorname{ext} C$ is separable and metrizable. Moreover, $\operatorname{ext} C$ admits an equivalent *complete* metric, since it is a G_δ in the compact set C. If E is infinite dimensional, then $\operatorname{ext} C$ is uncountable. We suspect that this is *all* that can be said, i.e., it is probably true that if X is an uncountable separable complete metric space, then X is homeomorphic to the set $\operatorname{ext} C$ (in the weak topology) for some convex body C in l_2. We have not been able to prove such a result, although we *can* show that if X is a *compact* uncountable metric space, then such a set C exists. The proof of this is complicated, and we present below a less general result which has a similar — but simpler — proof.

PROPOSITION 1.3. *Suppose that X is a compact metric space and that $X = X_1 \cup X_2$, where X_1 and X_2 are uncountable, closed and disjoint. Then there is a convex body C in a separable Hilbert space H such that X is homeomorphic to* $\operatorname{ext} C$ *(in the weak topology).*

Proof. Let $H = H_1 \oplus H_2$, where H_1, H_2 are copies of the space l_2. Since $C(X_1)$ is separable, a standard construction allows us to map X_1 homeomorphically into H_2: Choose a sequence $\{f_n\}_{n=1}^{\infty}$ which is dense in the unit sphere of $C(X_1)$ and define $\psi_1 : X_1 \to H_2$ by

$$\psi_1 x = (2, 8^{-1} f_1(x),\ 8^{-2} f_2(x), \cdots\ 8^{-n} f_n(x), \cdots), \qquad x \in X_1.$$

In addition to being a homeomorphism, ψ_1 has the property that $\| a - \psi_1 x \| < 1/4$ f $x \in X_1$, where $a = (2, 0, 0, \cdots) \in H_2$. We can define a similar map ψ_2 of X_2 into H_1, such that $\| b - \psi_2 x \| < 1/4$ if $b = (2, 0, 0, \cdots)$ and $x \in X_2$.

Since X_1 is an uncountable compact metric space, it contains a homeomorphic copy of the Cantor set [2, p. 445], which can itself be mapped continuously onto $[0,1]$. By Tietze's extension theorem, then, we can map X_1 continuously onto $[0,1]$. By the Hahn-Mazurkiewicz theorem, we can map $[0,1]$ continuously onto the weakly compact and weakly metrizable unit ball of H_1. Let ϕ_1 denote the composition of these two mappings, and let ϕ_2 denote the analogous mapping of X_2 onto the unit ball of H_2. We can define a map $F: X \to H_1 \oplus H_2$ as follows:

$$F(x) = (\phi_1 x, \psi_1 x) \qquad x \in X_1$$

$$F(x) = (\psi_2 x, \phi_2 x) \qquad x \in X_2.$$

The function F is clearly continuous, so it will be a homeomorphism if it is one-to-one. If $x, y \in X_1$ with $x \neq y$, then $\psi_1 x \neq \psi_1 y$ so $F(x) \neq F(y)$; similarly for points in X_2. If $x \in X_1$, $y \in X_2$, then the first coordinate of $\phi_1 x$ is at most 1, while that of $\psi_2 x$ is 2. Denote the closed convex hull of $F(X)$ by C; we must show that $\operatorname{ext} C = F(X)$ and that C has nonempty interior. Since $F(X)$ is compact, we know that $\operatorname{ext} C \subset F(X)$. To show that each point of $F(X)$ is extreme, suppose $x \in X_1$, say, and choose, by the (reformulated) Krein-Milman theorem (cf. e.g. [6]) a Borel probability measure μ on $F(X)$ which represents $F(x)$, i.e., which satisfies

$$\langle Fx, u \rangle = \int_{F(X)} \langle Fy, u \rangle \, d\mu(Fy), \qquad u \in H.$$

Since F is a homeomorphism, we can consider μ to be a measure on X and this formula becomes

$$(*) \qquad \langle Fx,u\rangle = \int_X \langle Fy,u\rangle\, d\mu(y), \qquad u\in H.$$

To show that Fx is extreme we must show that μ is the point mass at x. Now, if $u=(0,b/2)$, then $\langle Fy,u\rangle = 2$ for any y in X_1 and hence (*) becomes

$$2 = \int_{X_1} 2d\mu + \int_{X_2} \langle \phi_2 y, b/2\rangle\, d\mu \leqq 2\mu(X_1) + \mu(X_2)$$

since $\| \phi_2 y\| \leqq 1 = \| b/2\|$. Since $\mu(X_2) = 1-\mu(X_1)$ this implies that $\mu(X_2)=0$ and that μ is a probability measure on X_1. Now, our definition of ψ_1 shows that if u is evaluation at the $(n+1)-st$ coordinate in H_2 $(n \geqq 1)$, then

$$(Fx)_{n+1} = 8^{-n} f_n(x) = \int_{X_1} 8^{-n} f_n(y)\, d\mu(y).$$

Since the linear span of $\{f_n\}$ is dense in $C(X_1)$, we conclude that μ is the point mass at x.

We now complete the proof by showing that $p=(\frac{1}{2}a,\frac{1}{2}b)$ is in the interior of C. It is a consequence of the separation theorem that it suffices to prove that for any w in H of norm 1,

$$(**) \qquad \inf\langle C,w\rangle + 1/4 \leqq \langle p,w\rangle \leqq \sup\langle C,w\rangle - 1/4.$$

Writing $w=(u,v)$ $(u\in H_1,\ v\in H_2)$, we can take the (equivalent) norm $\|w\| = \max(\|u\|,\|v\|)$ on H, and thus assume that $\|u\| = 1 \geqq \|v\|$. Choose x_2 in X_2 such that $\phi_2(x_2) = 0$ and let $r = F(x_2) = (\psi_2(x_2),0)$. Since $\|\psi_2(x_2) - a\| < 1/4$, we have

$$|\langle r,w\rangle - \langle a,u\rangle| < 1/4.$$

Let $s=F(x_1)$ where $x_1\in X$ and $\langle \phi_1(x_1),u\rangle = \|u\| = 1$ and let $s' = F(x_1')$ where $\langle \phi_1(x_1'),u\rangle = -1$. Then we have

$$\begin{aligned}\langle s,w\rangle &= \langle \phi_1(x_1),u\rangle + \langle \psi_1(x_1),v\rangle = \\ &= 1 + \langle b,v\rangle + \langle \psi_1(x_1) - b,v\rangle \\ &\geqq 1 + \langle b,v\rangle - 1/4 = 3/4 + \langle b,v\rangle.\end{aligned}$$

Similarly, $\langle s',w\rangle \leqq -3/4 + \langle b,v\rangle$, so

$$\begin{aligned}\langle \tfrac{1}{2}(r+s),w\rangle &\geqq (1/2)[\langle a,u\rangle - 1/4 + 3/4 + \langle b,v\rangle] \\ &= \langle \tfrac{1}{2}(a+b),w\rangle + 1/4.\end{aligned}$$

Similarly, $\langle \frac{1}{2}(r+s'),w\rangle \leqq \langle \frac{1}{2}(a+b),w\rangle - 1/4$. Since $\frac{1}{2}(r+s)$, $\frac{1}{2}(r+s')$ are in C, this yields (**) and completes the proof.

It should be noted that G. Choquet has proved [1] that if X is a complete separable metric space, then X is homeomorphic to $\operatorname{ext} K$ for some compact convex metrizable simplex K in some locally convex space.

2. **Applications of Theorem** 1.1. Since any set of isolated points in a separable metric space is countable we have an immediate corollary to Corollary 1.2, which shows that a convex body in an infinite dimensional reflexive space cannot be too much like a polytope.

COROLLARY 2.1. *If C is a convex body in an infinite dimensional separable reflexive Banach space, then the extreme points of C are not isolated in the norm topology.*

Surprisingly the above corollary fails if we do not assume separability. This is shown below, but we first need a well-known lemma and a definition: A subset N of a normed linear space is called an *ε-net* if for some $\varepsilon > 0$, we have $\| x - y \| \geqq \varepsilon$ whenever x and y are distinct points of N.

LEMMA 2.2. *Suppose that E is a normed linear space and that N is a maximal ε-net in the unit sphere of E. Then the closed convex hull of N contains the ball at* 0 *of radius $1 - \varepsilon$.*

Proof. If the conclusion were to fail, we could find x in E and f in E^* with $\| x \| \leqq 1 - \varepsilon$, $\| f \| = 1$ and $f(x) > \sup f(N)$. For any $\delta > 0$ we could choose y in E, $\| y \| = 1$, such that $f(y) > 1 - \delta$. By maximality of N, there would exist z in N with $\varepsilon > \| y - z \| \geqq f(y) - f(z)$. Thus, $\sup f(N) \geqq f(z) > f(y) - \varepsilon > 1 - \delta - \varepsilon$ for any $\delta > 0$, so $1 - \varepsilon \leqq \sup f(N) < f(x) \leqq \| x \| \leqq 1 - \varepsilon$, a contradiction.

EXAMPLE 2.3. *Let H denote the Hilbert space which has Hilbert dimension equal to the power of the continuum $\aleph$ and let $0 < \varepsilon < 1/2$. Then there is a symmetric convex body C in the unit ball of H such that $\| x - y \| \geqq \varepsilon$ for any two distinct extreme points x and y of C.*

Proof. Choose an orthonormal basis for H of the form

$$\cup \{e_{\alpha,\eta} : \alpha \in A\}$$

where $\operatorname{card} A = \aleph$ and the union is taken over the set of all ordinals η with $\operatorname{card} \eta < \aleph$. For each η, let H_η be the closed linear span of $\{e_{\alpha,\eta} : \alpha \in A, \zeta \leqq \eta\}$. If $x \in H$, then x is in the closed linear span of at most countably many basis elements e_{α_n,η_n}, $n = 1, 2, \cdots$. If $\eta = \max \eta_n$, then $\operatorname{card} \eta < \aleph$ and $x \in H_\eta$. Thus, $H = \cup H_\eta$.

We will construct a certain maximal symmetric ε-net N in the unit sphere of H and take C to be the closed convex hull of N. (Note that "maximal symmetric" easily implies "maximal".) The ε-net N will be the union of symmetric maximal ε-nets N_η, where we define N_η in the unit sphere of each H_η by transfinite induction. If η is a limit ordinal, take N_η to be any maximal symmetric ε-net containing the (symmetric) ε-net

$$\cup \{N_{\eta'} : \eta' < \eta\}.$$

Assuming we have chosen N_η, construct $N_{\eta+1}$ as follows:

Let T be all points of the unit ball of H_η which have norm $> \varepsilon$, are in the weak closure of N_η, but are not in N_η. The set T is symmetric (since N_η is), so we can write $T = S \cup (-S)$, where S and $-S$ are disjoint and $S = \{x_i : i \in I\}$ for some index set I. The space H_η has Hilbert dimension $\aleph$, hence has cardinality $\aleph$. This implies that card $I \leqq \aleph$, so we can find a one-to-one map r from I into A. This defines a one-to-one map $x_i \to e_{r(i),\eta+1}$ from S into $\{e_{\alpha,\eta+1} : \alpha \in A\}$. (For simplicity of notation, we will drop the index $\eta + 1$.)

For each i in I let $\lambda_i \geqq 0$ be such that $1 = \| x_i + \lambda_i e_{r(i)} \|^2 = \| x_i \|^2 + \lambda_i^2$, and let

$$M = N_\eta \cup \{\pm[x_i \pm \lambda_i e_{r(i)}] : i \in I\}$$

where we take all possible choices of signs. The set M is obviously a symmetric subset of the unit sphere of $H_{\eta+1}$; we will show that it is an ε-net. First, note that any element x_i in S satisfies $\varepsilon^2 \leqq 2 - 2\| x_i \|^2$. Indeed, if y_β is a net in N_η converging weakly to x_i, then for $y_\beta \neq y_\gamma$ we have

$$\varepsilon^2 \leqq \| y_\beta - y_\gamma \|^2 = 2 - 2(y_\beta, y_\gamma).$$

Taking the weak limit first over γ then over β yields $\varepsilon^2 \leqq 2 - 2(x_i, x_i) = 2 - 2\|x_i\|^2$, which was to be proved. To see that M is an ε-net, suppose that u and v are distinct elements of M; we must show that $\| u - v \| \geqq \varepsilon$. There are four cases to consider:

I. $u = x_i + \lambda_i e_{r(i)}$, $v = \pm x_i - \lambda_i e_{r(i)}$. In this case,

$$\| u - v \|^2 = \| x_i \pm x_i \|^2 + 4\lambda_i^2 \geqq 4\lambda_i^2 = 4(1 - \| x_i \|^2) \geqq 2\varepsilon^2.$$

II. $u = x_i + \lambda_i e_{r(i)}$, $v = -x_i + \lambda_i e_{r(i)}$. Here,

$$\| u - v \| = 2\| x_i \| > 2\varepsilon.$$

III. $u \in N_\eta$, $v = x_i \pm \lambda_i e_{r(i)}$. Since $e_{r(i)} \perp H_\eta$, we have

$$\begin{aligned} \| u - v \|^2 &= \| u - x_i \|^2 + \| x_i - v \|^2 = 1 - 2(u, x_i) + \| x_i \|^2 + \lambda_i^2 \\ &= 2 - 2(u, x_i). \end{aligned}$$

We can choose a net $\{y_\beta\}$ in N_η converging weakly to x_i, with $y_\beta \neq u$. Thus,

$$2 - 2(u, x_i) = \lim[2 - 2(u, y_\beta)] = \lim \| u - y_\beta \|^2 \geqq \varepsilon^2.$$

IV. $u = x_i \pm \lambda_i e_{r(i)}$, $v = x_j \pm \lambda_j e_{r(j)}$, $i \neq j$. The vectors $\pm \lambda_i e_{r(i)}$, $\pm \lambda_j e_{r(j)}$ and $x_i - x_j$ are pairwise orthogonal, so

$$\begin{aligned} \| u - v \|^2 &= \| u - x_i \|^2 + \| x_i - x_j \|^2 + \| x - v \|^2 \geqq \lambda_i^2 + \lambda_j^2 \\ &\geqq \varepsilon^2/2 + \varepsilon^2/2. \end{aligned}$$

This shows that M is a symmetric ε-net, and we can choose a maximal symmetric ε-net $N_{\eta+1}$ in the unit sphere of $H_{\eta+1}$ which contains M; this completes the induction.

Let $N = \cup N_\eta$; it is easily verified that N is a maximal symmetric ε-net in the unit sphere of H. Let $C = \mathrm{cl\,conv}\, N$; by Lemma 2.2 C has nonempty interior. Each point of N is extreme (even "strongly exposed," see the definition preceding Proposition 2.6) in the unit ball of H, hence has the same property in C. We must show that C has no other extreme points. If $x \in \mathrm{ext}\, C$, $x \notin N$, then x is in the weak closure of N, but not in $\mathrm{int}\, C$. By Lemma 2.2 again, $\| x \| \geqq 1-\varepsilon$ and since $\varepsilon < 1/2$, $\| x \| > \varepsilon$. Since $x \in C$ it is the limit of a sequence $\{x_n\}$ of convex combinations of elements of $N = \cup N_\eta$. Each x_n is in the convex hull of some N_{η_n}, so if $\eta = \max \eta_n$, then x is in the closed convex hull C_η of N_η. Clearly, x is extreme in C_η, so it is in the weak closure of N_η. Our construction shows that x is a nontrivial convex combination of two elements in $N_{\eta+1}$, a contradiction which completes the proof.

We now present some further applications of Theorem 1.1.

A normed linear space E is said to have the 3.2 *intersection property* if it satisfies the following:

Whenever B_1, B_2 and B_3 are closed balls in E and $B_i \cap B_j$ is nonempty for each i and j, then $B_1 \cap B_2 \cap B_3$ is nonempty. The Banach spaces with this property have been studied in [3]. They are useful in the study of extensions of operators, and (as shown in [3, Ch. 4]) they form a class of spaces which contains the $C(X)$ and $L_1(\mu)$ spaces and is closed under l_1 or l_∞ direct sums (of arbitrary cardinality).

COROLLARY 2.4. *If E is an infinite dimensional reflexive Banach space, then E does not have the* 3,2 *intersection property.*

Proof. It is proved in [3, Theorem 4.7] that the 3,2 intersection property implies that $\| x - y \| = 2$ whenever x and y are distinct extreme points of the unit ball of E. In [3, Theorem 4.4] (and the subsequent remark) it is shown that any infinite dimensional Banach space with the 3,2 intersection property contains a separable infinite dimensional subspace with the same property. The proof is now an immediate consequence of Corollary 2.1.

Suppose that E is a closed subspace of the Banach space F. A map ψ from E^* to F^* is a *continuous norm preserving extension map* [3] provided it is continuous in the norm topologies and satisfies

$$\| \psi f \| = \| f \| \text{ and } \psi f \text{ extends } f, \text{ for each } f \text{ in } E^*.$$

It is known [3] that a finite dimensional space E has the property that such a map exists for *every* $F \supset E$ if and only if the unit ball U of E is a polyhedron, i.e. if and only if $\mathrm{ext}\, U$ is isolated. In infinite dimensional spaces, the above property implies that $\mathrm{ext}\, U$ is isolated [3], but Example 3.2, combined with the following corollary, shows that the reverse implication fails.

COROLLARY 2.5. *Suppose that E is an infinite dimensional reflexive Banach space. Then there exists a Banach space $F \supset E$ for which there is no continuous norm preserving extension map from E^* to F^*.*

Proof. Suppose, to the contrary, that such a map exists for each $F \supset E$. We need only show that there is an infinite dimensional separable closed subspace M of E which has the same property. Indeed, from [3, p. 94] we know that this property implies that the extreme points of the unit ball of M^* are isolated, and Corollary 2.1 yields a contradiction. Now, by [5, Proposition 1] there exists (in any reflexive space E) a separable infinite dimensional closed subspace M of E and a projection of norm 1 from E onto M. By the same argument as was used in [3, pp. 88–89] it follows that M has the desired property.

(It follows from the proof of Theorem 7.6 of [3] that the above space F can be chosen to satisfy $\dim F/E = 1$.)

A point x in a convex subset C of a topological vector space E is said to be an *exposed point* of C if there exists f in E^* such that $f(x) = \sup f(C) > f(y)$ whenever $y \in C$, $y \neq x$. Obviously, every exposed point of C is an extreme point. This leads to the following problem:

PROBLEM. *Can the unit ball of an infinite dimensional reflexive Banach space have countably many exposed points?*

It follows from [5, Corollary 1] that the exposed points of the ball are always weakly dense in the extreme points. Branko Grünbaum has shown (oral communication) that there exists a three dimensional space with unit ball U such that $\operatorname{ext} U$ is uncountable but the set of exposed points of U is countable.

It is known [4] that in a separable Banach space, every weakly compact convex set C is the closed convex hull of the set $\operatorname{str\,exp} C$ of its strongly exposed points. (A point x in a convex set C is *strongly exposed* provided there exists f in E^* with $f(x) = \sup f(C)$ such that $f(x_n) \to \sup f(C)$, $x_n \in C$, implies $\| x - x_n \| \to 0$.) Thus, such points are weakly dense in the set $\operatorname{ext} C$.

PROPOSITION 2.6. *If E is an infinite dimensional separable reflexive Banach space, then there exists a symmetric convex body U in E which has countably many strongly exposed points.*

Proof. Choose $0 < \varepsilon < 1$ and let N be a maximal symmetric ε-net in the unit sphere of E. By Lemma 2.2, the closed convex hull U of N has nonempty interior, hence is a symmetric convex body. It follows easily from the definition of strongly exposed point (and the fact that N is norm closed) that $\operatorname{str\,exp} U \subset N$. As noted above, U is the closed convex hull of $\operatorname{str\,exp} U$, so the latter is infinite and (since E is separable and N is norm isolated) countable.

Note that even though $\operatorname{str\,exp} U$ is countable, its weak closure contains the uncountable set $\operatorname{ext} U$.

3. Dual unit balls with countably many extreme points.

THEOREM 3.1. *Suppose that E is a normed linear space and that the unit ball U^* of E^* has countably many extreme points. Then*

(i) *E^* is separable and*
(ii) *E contains no infinite dimensional reflexive subspace.*

Proof. (i) Since U^* is weak* compact and $\operatorname{ext} U^* = \{f_n\}_{n=1}^{\infty}$ is an F_σ, the Choquet-Bishop-de Leeuw theorem [6, esp. p. 30] shows that for each f in U^* there is a probability measure μ on $\operatorname{ext} U^*$ (not necessarily unique) such that

$$f(x) = \int_{\operatorname{ext} U^*} g(x) d\mu(g) \quad \text{for each } x \text{ in } E.$$

Letting $\mu_n = \mu(f_n)$, we have $\mu_n \geqq 0$, $\sum \mu_n = 1$, and

$$f(x) = \sum \mu_n f_n(x) \text{ for each } x \text{ in } E.$$

If S is the set of all sequences $\{\mu_n\}$ with $\mu_n \geqq 0$ and $\sum \mu_n = 1$, then $S \subset l_1$ and for any $\{\lambda_n\}$ in S, $g = \sum \lambda_n f_n$ defines a member of U^*. Thus, we have defined a map from the norm-separable space S onto U^*; since

$$\| f - g \| = \sup \{ | f(x) - g(x) | : \| x \| \leqq 1\} \leqq \sum | \mu_n - \lambda_n |,$$

the map is norm-to-norm continuous and hence U^* is norm separable, which implies the same for E^*.

(ii) If F is an infinite dimensional reflexive subspace of E, then the set $\operatorname{ext} U_F^*$ of extreme points of the unit ball of F^* is (by Theorem 1.1) uncountable. But each f in $\operatorname{ext} U_F^*$ can be extended (by applying the Krein-Milman theorem to the convex set of all its norm one extensions) to an extreme point of U^*, and this implies that $\operatorname{ext} U^*$ is uncountable.

The above result indicates that if $\operatorname{ext} U^*$ is countable, then E^* is very much like l_1. In fact, the only examples we know (of space E^* for which $\operatorname{ext} U^*$ is countable) are isomorphic to the l_1-direct sum of a sequence of finite dimensional spaces. This suggests the following problem.

Problem. Suppose that E is a Banach space and that $\operatorname{ext} U^*$ is countable. Do weak and norm convergence coincide for sequences in E^*?

The proof of the next corollary was suggested to us by Professor Harry Corson.

Corollary 3.2. *If K is a compact convex subset of a locally convex space and if* $\operatorname{ext} K$ *is countable, then K is metrizable.*

Proof. Let E denote the sup-normed space of all continuous real-valued affine functions on K. The evaluation map of K into E^* is an affine homeomorphism (in the weak* topology) of K onto a subset (which we denote by K) of the unit ball U^* of E^*. It is readily verified that U^* is the convex hull of $K \cup (-K)$ and hence $\operatorname{ext} U^*$ is countable, so that E^* is separable. In particular, E is separable, so U^* is weak* metrizable, and the same is true for K.

The above corollary is a special case of a more general result which we had previously obtained by a different method. Much more general results have been communicated to us by both H. Corson and G. Choquet; the most general of these appears to be the following: (Choquet) If K is compact convex and if ext K is the continuous image of a complete separable metric space, then K is metrizable.

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